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The stability of oscillatory internal waves

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The stability of a periodic internal wave has been investigated experimentally and theoretically. From the analysis it is found that if a primary wave, with wave-number k_0 and frequency ω_0, is perturbed by two infinitesimal wave-like disturbances with wave-numbers k_1 and $k_1 + k_0$ and frequencies ω_1 and $\omega_1 + \omega_0$, exponential growth of these disturbances will take place under certain conditions. The analysis also indicates which resonantly interacting disturbances can induce an instability and, when viscous dissipation is accounted for, predicts the minimum amplitude for which a wave is unstable. Experimental results demonstrate that this type of instability can cause the breakdown of a first mode internal wave propagating in a fluid composed of two layers of uniform density separated by a thin region in which the density varies continuously.

1. Introduction

One of the basic unanswered questions about progressive oscillatory internal waves concerns the mechanism by which they 'break', a phenomenon which is of considerable importance to the understanding of vertical transport processes in the oceans (Munk 1966). Owing to the dearth of experimental data on this subject, however, this question of 'break-up' has received to date only conjectural answers, a case in point being Phillips's (1966) proposal that the probable mechanism of breaking is shear induced, in the sense that an internal wave will break when the shear associated with it becomes sufficiently large to promote a local dynamic instability.

During the course of an experimental study of internal waves in a stratified fluid composed of two miscible fluid layers, Keulegan & Carpenter (1961) observed that, under certain circumstances, these waves generated 'vortices' in the region separating the two layers. Although unable to determine the exact nature of these disturbances, Keulegan & Carpenter did report that the 'vortices' appeared only when the thickness of the region of varying density between the layers became larger than some critical value, an observation which is in direct opposition to what would be expected if the phenomenon were due to shear instability. This is so because, as Munk (1966) has shown, for a fixed amplitude and frequency, the local Richardson number, and hence the stability, increase with the thickness of the interfacial region. This apparent inconsistency, although

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far from conclusive, does suggest, therefore, that there may exist a mechanism, other than that of shear instability, by which internal waves spontaneously destroy themselves.

In an attempt to determine the cause of this 'breaking' phenomenon, we began an experimental program aimed at studying qualitatively the lowest mode periodic internal wave in a two-layer fluid, and observed that a single train of such waves is indeed subject to some form of instability which appears to be different from the shear instability mechanism proposed by Phillips. The purpose of this paper will be then to present a new theoretical investigation of the stability of internal waves to a particular type of disturbance, and to demonstrate that this new theory does, in fact, explain our experimental observations. In particular, it will be shown that a single train of oscillatory internal waves can be unstable to infinitesimal disturbances made up of free progressive waves with amplitudes that vary slowly in time. The mechanism of the instability will be similar, in a sense, to the weak resonant interactions which have been shown (McGoldrick, Phillips, Huang & Hodgson 1966) to produce energy transfer between surface waves and which have been investigated theoretically by Thorpe (1966) in reference to internal wave motion. But, in both objective and result, these previous studies differ significantly from our investigation, which will concern the fate of infinitesimal waves that resonantly interact with a single finite amplitude wave. In short, it will be seen that interactions of this type can be divided into two classes, depending on the manner in which the amplitudes of the disturbances vary with time, and that for one class of these interactions the amplitudes will always increase at an exponential rate, thus reaching an appreciable size no matter how small the initial disturbances.

The theoretical problem is now formulated in terms of the two-dimensional equations of motion for an incompressible, inviscid and non-diffusive fluid of variable density

$$\hat{\rho}\frac{D\hat{u}}{D\hat{t}}+\frac{\partial \hat{P}}{\partial \hat{x}} = 0, \quad \hat{\rho}\frac{D\hat{v}}{D\hat{t}}+\frac{\partial \hat{P}}{\partial \hat{y}}+g\hat{\rho} = 0,$$

$$\frac{\partial \hat{u}}{\partial \hat{x}}+\frac{\partial \hat{v}}{\partial \hat{y}} = 0, \quad \frac{D\hat{\rho}}{D\hat{t}} = 0,$$

where $\hat{y}$ is directed upwards, $\hat{u}$ and $\hat{v}$ are fluid velocities, $\hat{P}$ is the pressure and $D/D\hat{t}$ denotes the substantive time derivative moving with a fluid particle. Using the familiar method of cross-differentiation the x and y momentum equations can now be combined into a single equation not involving the pressure. Then introducing L, a characteristic length, the quantities

$$\delta = \tfrac{1}{2}\ln(\hat{\rho}_{\max}/\hat{\rho}_{\min}), \quad \rho = \frac{1}{\delta}\ln\{\hat{\rho}/(\hat{\rho}_{\max}\,\hat{\rho}_{\min})^{\frac{1}{2}}\},$$

$$(x,y) = (\hat{x},\hat{y})/L, \quad \Omega = (g\delta/L)^{\frac{1}{2}}, \quad t = \Omega\hat{t}$$

and the dimensionless streamfunction ψ, defined by

$$\hat{u} = \Omega L\frac{\partial\psi}{\partial y}, \quad \hat{v} = -\Omega L\frac{\partial\psi}{\partial x},$$

we arrive at the dimensionless equation of motion

$$\frac{D}{Dt}(\nabla^2\psi)+\delta\frac{\partial\rho}{\partial y}\frac{D}{Dt}\left(\frac{\partial\psi}{\partial y}\right)+\delta\frac{\partial\rho}{\partial x}\frac{D}{Dt}\left(\frac{\partial\psi}{\partial x}\right)=\frac{\partial\rho}{\partial x}.$$

Restricting our interest to fluids for which $\delta \ll 1$ and making use of a modified Boussinesq approximation, we now neglect the terms which involve δ explicitly, thus arriving at the simplified equation of motion

$$\frac{D}{Dt}\nabla^2\psi=\frac{\partial\rho}{\partial x},$$

or equivalently

$$\frac{\partial}{\partial t}\frac{D}{Dt}(\nabla^2\psi)=\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\frac{\partial\rho}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\rho}{\partial x}\right),\quad \frac{D\rho}{Dt}=0, \tag{1}$$

the equations which will serve as the basis for the subsequent theoretical analysis.

In what follows, we shall obtain an approximate solution of equations (1) which will describe a triad of small amplitude wave-trains that are weakly coupled through resonant interaction. The point of interest in this development will be the energy exchange between the different members of the triad, and it will be shown that, as a consequence of this exchange, a single train of periodic waves is unstable to particular infinitesimal disturbances. This resonant interaction induced instability is, in some respects, similar to the instability of the Stokes wave recently investigated by Benjamin & Feir (1967), but, as will be seen, the interaction between internal waves is much more direct than that between surface gravity waves.

In a later section we shall describe the experiments which motivated this study. Here the strength of the internal wave interaction is demonstrated by the vigour with which the resonant instability is manifested. Indeed, the phenomenon is so dramatic that it is easily observed without the need of any particularly careful measurements such as those that are required to study surface gravity wave interactions (McGoldrick *et al.* 1966).

Finally, by accounting for the influence of viscosity, we shall show that the instability can occur only when the primary wave amplitude exceeds some critical value which can easily be determined theoretically.

2. Analysis

We shall develop here an approximate solution to (1) representing a triad of small amplitude internal wave-trains. The analysis will make use of techniques familiar in the study of wave–wave interactions (for example see McGoldrick 1965); namely, the relevant parameters will be expanded in powers of the small amplitudes of the different waves and these amplitudes will, in turn, be allowed to be weak functions of time. Thus we express

$$\psi=\sum_{i=0}^{2}a_i\psi_i+\sum_{i=0}^{2}\sum_{j=i}^{2}a_ia_j\psi_{ij}+O(a^3),$$

$$\rho=-\int R(y)\,dy+\sum_{i=0}^{2}a_i\rho_i+O(a^2),$$

and

$$da_i/dt=O(a^2).$$

Substituting these into (1) and collecting terms of $O(a)$ leads to

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi_i + R\frac{\partial^2\psi_i}{\partial x^2} = 0, \tag{2}$$

$$\frac{\partial\rho_i}{\partial t} + R\frac{\partial\psi_i}{\partial x} = 0, \tag{3}$$

which form the basis of the well-known theory of infinitesimal internal waves (for example see Yih 1960). We note here that (2) has a solution of the form

$$\psi_i = f_i(y)\sin(kx+\omega t),$$

where f_i satisfies

$$\frac{d^2f}{dy^2} + \left(\frac{k^2}{\omega^2}R - k^2\right)f = 0, \tag{4}$$

with boundary conditions

$$f = 0 \quad \text{at} \quad y = D_1, D_2,$$

while, from (3),

$$\rho^{(1)} = -(k/\omega)Rf\sin(kx+\omega t).$$

Clearly, system (4), being of the Sturm–Liouville type, has an infinite number of solutions f for any given k, each solution corresponding to a different eigenvalue $\gamma^2 \equiv (k/\omega)^2$. Although these can be developed, in principle, for arbitrary $R(y)$, it suffices for our purposes to consider only the two particular cases $R = 1$ and $R = \operatorname{sech}^2 y$.

For $R = 1$, $D_1 = -1$ and $D_2 = 1$, the solutions are simply

$$f = \sin\tfrac{1}{2}m\pi(y+1), \quad \gamma^2 = (\tfrac{1}{2}m\pi)^2 + k^2 \quad \text{for} \quad m = 1, 2, \ldots, \tag{5}$$

where each value of m corresponds to a different wave mode and where the dimensionless propagation velocity $1/\gamma$ decreases in absolute magnitude as the mode number increases.

Similarly, if $R = \operatorname{sech}^2 y$ and the conditions that f vanish are placed at plus and minus infinity, closed form solutions can also be found (Groen 1948). For future reference, we note here those corresponding to the first three modes,

$$\left.\begin{aligned}
&f = \operatorname{sech}^{|k|}y, \quad \gamma^2 = |k|(1+|k|) \quad \text{for} \quad m = 1,\\
&f = \operatorname{sech}^{|k|}y\tanh y, \quad \gamma^2 = (1+|k|)(2+|k|) \quad \text{for} \quad m = 2,\\
&f = \operatorname{sech}^{|k|}y\left(1+\frac{1{\cdot}5+|k|}{1+|k|}\operatorname{sech}^2 y\right), \quad \gamma^2 = (2+|k|)(3+|k|) \quad \text{for} \quad m = 3,
\end{aligned}\right\} \tag{6}$$

and present in figure 1 the lines of constant density as they are distorted by the first and second mode waves of this type.

Now turning our attention to the determination of ψ_{ij}, we substitute ψ_i into the $O(a^2)$ terms of (1) and obtain

$$\sum_{i=0}^{2}\sum_{j=i}^{2} a_i a_j\left\{\frac{\partial^2}{\partial t^2}\nabla^2\psi_{ij} + R\frac{\partial^2\psi_{ij}}{\partial x^2} - \sum_{\sigma=\pm1} Z_{ij\sigma}(y)\cos([k_i+\sigma k_j]x + [\omega_i+\sigma\omega_j]t)\right\} - 2\sum_{i=0}^{2}\frac{da_i}{dt}\frac{k_i^2}{\omega_i^3}Rf_i\cos(k_i x+\omega_i t) = 0, \tag{7}$$

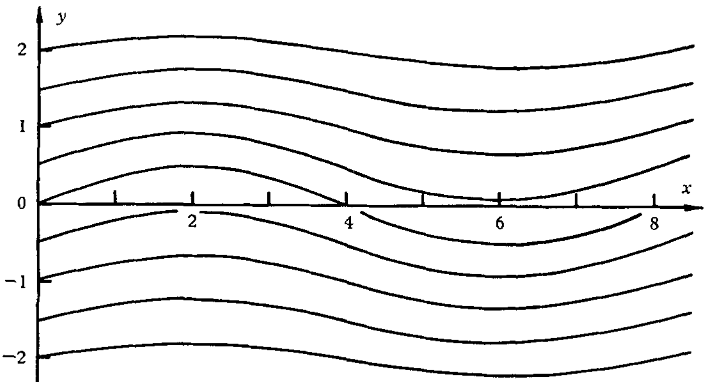
where

$$Z_{ij\sigma} = \frac{\sigma}{2}(\omega_i+\sigma\omega_j)\left\{\frac{k_i}{\omega_i}\left(\frac{k_i}{\omega_i}+\frac{k_i+\sigma k_j}{\omega_i+\sigma\omega_j}\right)\left(k_i\frac{df_j}{dy}Rf_i - \sigma k_j f_j\frac{dRf_i}{dy}\right) + \frac{k_j}{\omega_j}\left(\frac{k_j}{\omega_j}+\frac{k_i+\sigma k_j}{\omega_i+\sigma\omega_j}\right)\left(\sigma k_j f_j R\frac{df_i}{dy} - k_i f_i\frac{dRf_j}{dy}\right)\right\}. \tag{8}$$

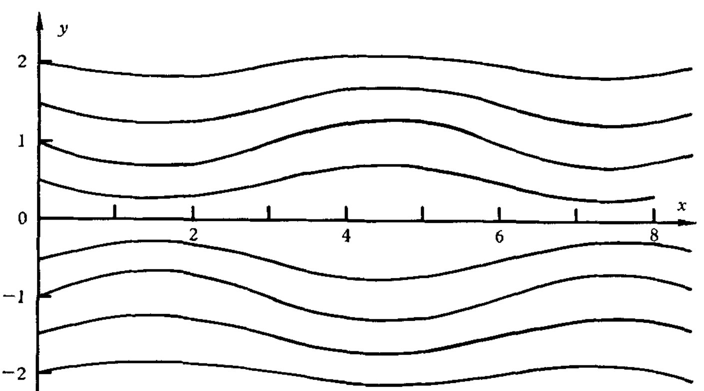
We are interested here in the case of three waves which are coupled through a resonant interaction, that is, where

$$k_0+k_1 = k_2 \quad \text{and} \quad \omega_0+\omega_1 = \omega_2. \tag{9}$$

When this resonance condition is met, (7) can be solved only for specific values of the amplitude variations, da_i/dt. To see this we need only consider the determination of the components $\psi_{0,1}$, $\psi_{0,2}$ and $\psi_{1,2}$ which arise as a result of the inter-



(*a*)



(*b*)

FIGURE 1. (*a*) Lines of constant density for a first mode wave in a two-layer fluid. (*b*) Lines of constant density for a second mode wave in a two-layer fluid.

action between the different wave-trains. For example, to construct $\psi_{0,1}$ we substitute the form $\psi_{0,1} = g(y)\cos(k_2x+\omega_2t)$ into (7), obtaining

$$\frac{d^2g}{dy^2}+\left(R\frac{k_2^2}{\omega_2^2}-k_2^2\right)g = \frac{Z_{0,1,1}}{\omega_2^2}+2\frac{k_2^2}{\omega_2^3}Rf_2\alpha_2,$$

where $da_2/dt \equiv \alpha_2 a_1 a_0$. However, since k_2, ω_2 are eigenvalues of the homogeneous counterpart of this equation, a solution exists only if the right-hand side is orthogonal to the homogeneous solution, f_2. This then requires that

$$\alpha_2 = -\frac{1}{2}\frac{\omega_2}{k_2^2}\int_{D_1}^{D_2} Z_{0,1,1}f_2\,dy\bigg/\int_{D_1}^{D_2} Rf_2^2\,dy. \tag{10a}$$

Similarly, in order to determine $\psi_{0,2}$ and $\psi_{1,2}$, we require that

$$da_0/dt = \alpha_0 a_1 a_2, \quad da_1/dt = \alpha_1 a_0 a_2,$$

and

$$\alpha_0 = -\frac{1}{2}\frac{\omega_0}{k_0^2}\int_{D_1}^{D_2} Z_{1,2,-1} f_0 \, dy \Big/ \int_{D_1}^{D_2} R f_0^2 \, dy, \tag{10b}$$

$$\alpha_1 = -\frac{1}{2}\frac{\omega_1}{k_1^2}\int_{D_1}^{D_2} Z_{0,2,-1} f_1 \, dy \Big/ \int_{D_1}^{D_2} R f_1^2 \, dy. \tag{10c}$$

The amplitude equations

$$da_0/dt = \alpha_0 a_1 a_2, \quad da_1/dt = \alpha_1 a_0 a_2, \quad da_2/dt = \alpha_2 a_0 a_1, \tag{11}$$

together with equations (10) defining the constants α_i, describe then the energy interchange between the three components of the resonant triad of waves. These equations are general to all wave–wave interactions which are of second order in the wave amplitudes and were obtained, for example, by McGoldrick (1965) in studying the interaction of gravity–capillary surface waves.

The integration of these amplitude equations is easily accomplished in terms of Jacobi elliptic functions. For the case $\alpha_1\alpha_2 < 0$ and $a_2 = 0$ at $t = 0$, the appropriate solution is

$$\left.\begin{aligned} a_0 &= a_0(0)\,\mathrm{dn}\,(z, l), \quad a_1 = a_1(0)\,\mathrm{cn}\,(z, l),\\ a_2 &= -a_1(0)\,(-\alpha_2/\alpha_1)^{\frac{1}{2}}\,\mathrm{sn}\,(z, l), \end{aligned}\right\} \tag{12}$$

where $z = a_0(0)\,(-\alpha_2\alpha_1)^{\frac{1}{2}} t$, $l = -(\alpha_0 a_1^2(0))/(\alpha_1 a_0^2(0))$ and sn, cn and dn are Jacobi elliptic functions in the notation of Milne-Thompson (1964). Similarly, the solution for $\alpha_1\alpha_2 > 0$ is

$$\left.\begin{aligned} a_0 &= a_0(0)\,\mathrm{dc}\,(\tilde{z}, \tilde{l}), \quad a_1 = a_1(0)\,\mathrm{nc}\,(\tilde{z}, \tilde{l}),\\ a_2 &= a_1(0)\,(\alpha_2/\alpha_1)^{\frac{1}{2}}\,\mathrm{sc}\,(\tilde{z}, \tilde{l}), \end{aligned}\right\} \tag{13}$$

where

$$\tilde{z} = a_1(0)\,(\alpha_1\alpha_2)^{\frac{1}{2}} t \quad \text{and} \quad \tilde{l} = 1 - l.$$

Although these solutions are well known (for example McGoldrick reported the solution for $\alpha_1\alpha_2 < 0$) it seems that previous workers have overlooked one very interesting consequence of the resulting amplitude variations, namely the fact that under certain circumstances a single wave is unstable to an infinitesimal disturbance consisting of a resonantly interacting wave. To see this, let us consider a single wave, with amplitude $a_0 = A$, on which we impose a disturbance consisting of two additional waves with initial amplitudes $a_1(0)$, $a_2(0) \ll A$. If these three waves form a resonant triad, then the resulting amplitude variations will be described by the linearized equivalents of equations (9),

$$da_1/dt = \alpha_1 A a_2, \quad da_2/dt = \alpha_2 A a_1, \tag{14}$$

which have solutions of the form

$$a_1, a_2 \sim \exp\{\pm A\sqrt{(\alpha_1\alpha_2)}\,t\}.$$

Clearly, the nature of these solutions is highly dependent on the sign of the product $\alpha_1\alpha_2$; in fact, if $\alpha_1\alpha_2 > 0$, we can say that the primary wave is unstable to this particular disturbance. This result can, of course, be deduced directly from the exact solutions (12) and (13) by allowing $a_1(0)$, and hence l, to approach zero.

This resonant instability is the phenomenon of interest here, and, hence, it is worth while to summarize the conditions under which a primary wave, (k_0, ω_0), is unstable. First, there must exist a pair of free waves forming a resonant triad with the primary wave, that is there must exist eigenpairs (ω_1, k_1) and (ω_2, k_2) which satisfy the resonance condition (9). And, secondly, the product of $\alpha_1\alpha_2$ computed from (10) must be positive. If these conditions are met, our theory would predict that an infinitesimal disturbance initially composed of either of the resonant free waves will grow at an exponential rate until the disturbance amplitude becomes comparable to the primary wave, or until the disturbance itself becomes unstable through some other resonant interaction.

In the following section we shall apply these criteria to wave motion in some specific stratified fluids and shall find that, in each case considered, an infinite number of disturbances exist which can destabilize a given wave. In addition, in a later section, we shall present experimental evidence to show that this type of instability does occur and that the destabilizing disturbance is one of those predicted by the analysis.

3. Examples

Having obtained the criteria for determining whether or not a single train of internal waves is unstable to a certain infinitesimal disturbance, we can proceed to apply them to waves propagating in particular stratified fluids. Although the task involves rather complicated calculations, owing to the nonlinear nature of the dispersion relation and the fact that there are an infinite number of intermode combinations which may induce an instability, we can derive some useful approximate results by restricting our interest to low-frequency waves in simple stratifications. The questions to be answered here are: first, which resonantly interacting disturbances can induce instability;† and, secondly, what is the rate at which these disturbances grow?

For purposes of comparing the predictions of the theoretical analysis with the experimental results of the next section, we shall choose as one example the case of a low-frequency first mode wave propagating in a two-layer fluid of depth $2D$ in which the density gradient at rest is given by $R = \operatorname{sech}^2 y$. In addition, we shall investigate the case of an exponentially stratified fluid (for which $R = 1$) contained between walls at $y = \pm 1$, since this will allow us to obtain some simple results regarding the stability of low-frequency primary waves of arbitrary mode.

Consider first the case of $R = \operatorname{sech}^2 y$. For a two-layer fluid of finite depth it is not possible to use directly the results presented in (6) for the eigenfunctions, $f(y)$, which were derived for an infinite depth. However, because our interest will be restricted to small wave-numbers and large depths ($D \gg 1$), an approximate solution of the eigenvalue problem (4) can easily be found by making use

† It has recently come to our attention that this first question can be answered without detailed calculation by making use of a general stability criterion developed by Hasselmann (1967).

of the method of singular perturbation expansions. Thus, in the 'inner' region we have

$$\frac{d^2f}{dy^2}+\gamma^2\,\mathrm{sech}^2\,yf = k^2f,$$

while, in the 'outer' regions and in terms of the appropriate stretched co-ordinate $\tilde{y} = ky$, equation (4) reduces to

$$\frac{d^2\tilde{f}}{d\tilde{y}^2}-\tilde{f} = 0.$$

Applying the conditions that $\tilde{f} = 0$ at $y = \pm D$, and matching the two solutions in the overlap region in the usual way, we easily find that: in the inner region,

$$f(y) = P_{m-1}(\eta)+k\coth kD\left\{Q_{m-1}(\eta)+(2m-1)\int_{-1}^{\eta}[P_{m-1}(\eta)P_{m-1}(x)Q_{m-1}(x) - Q_{m-1}(\eta)P^2_{m-1}(x)]\,dx\right\}+O(k^2),$$

where P_n and Q_n are, respectively, the nth Legendre functions of the first and second kind, $\eta = \tanh y$, and

$$\gamma^2 = m(m-1)+(2m-1)k\coth kD+O(k^2), \tag{15}$$

while, in the outer regions,

$$\tilde{f} = \frac{\sinh k(D-y)}{\sinh kD}+O(k) \quad \text{for} \quad y > 0,$$

$$\tilde{f} = (-1)^{m-1}\frac{\sinh k(D+y)}{\sinh kD}+O(k) \quad \text{for} \quad y < 0.$$

By comparing these results with (6) it can be seen that they agree with the exact solution when $D\to\infty$.

With these approximate expressions for $f(y)$ and $\gamma^2(k)$, it is not difficult now to compute from (9) the mode numbers, frequencies and wave-numbers (m_1, m_2; ω_1, ω_2; k_1, k_2) of all disturbances which interact resonantly with a first mode wave of frequency ω_0 and wave-number k_0. Then, for each of these disturbances, we can evaluate from (10) the product $\alpha_1\alpha_2$ which determines not only the absence or presence of an instability, but also, in the latter case, its rate of growth. The results of these calculations (the details of all the calculations presented in this section are given by Davis 1967) indicate that, up to $O(k_0^2)$, positive values of $\alpha_1\alpha_2$ are possible only if component 1 is directed opposite to both the primary wave and component 2; and, if both mode numbers m_1 and m_2 are greater than 1 with $m_2 = m_1 \pm 1$. Then

$$\alpha_1\alpha_2 = k_0^2\gamma_1\gamma_2(\gamma_1+\gamma_2)^2/(64m^2-16), \tag{16}$$

where m is the smaller of m_1 and m_2, and

$$\omega_1 = -\omega_0\Big/\left[1+\left(\frac{m_1^2-m_1}{m_2^2-m_2}\right)^{\frac{1}{2}}\right]+O(\omega_0^2).$$

Let us now consider a primary wave of arbitrary mode propagating in an exponentially stratified fluid of finite depth. Although the simple form of the appropriate eigenfunctions, $f(y)$, (see equation (5)) greatly reduces the computational difficulty, it is still necessary to restrict interest to long waves for which the dispersion relation can be simplified to $\gamma_i = \pm\frac{1}{2}\pi m_i + O(k^2)$. Under this restriction, it is possible to show that, up to $O(k_0^2)$, disturbances which produce positive values of $\alpha_1\alpha_2$ are the following.

(*a*) Both disturbances propagate in the same direction as the primary wave, and $m_2 = m_0 + m_1$, with $m_0 > m_1$. Then,

$$\alpha_1\alpha_2 = \pi^2 k_0^2 m_1(m_1+m_0)^2(m_0-m_1)/64m_0^2 + O(k_0^3), \tag{17a}$$

and

$$\omega_1 = -m_1\omega_0/m_0 + O(\omega_0^2).$$

(*b*) Disturbance component 1 travels opposite to the primary wave and component 2, and $m_2 = m_0 + m_1$. Then,

$$\alpha_1\alpha_2 = \pi^2 k_0^2 m_0^2 m_1(m_1+m_0)/(128m_1^2 + 64m_0),$$

and

$$\omega_1 = -m_1\omega_0/(2m_1+m_0). \tag{17b}$$

As can be seen from the foregoing, for every possible primary wave there are an infinite number of disturbance pairs which induce instabilities and transfer energy from the primary wave to waves having different modes, different frequencies, and even a different propagation direction.

Clearly, owing to the vast number of combinations of possible primary waves and destabilizing disturbances, it is impossible to verify experimentally all the results of this section. In the subsequent section we shall present, therefore, experimental investigation of the stability of a particular internal wave, which will indicate that, as the analysis predicts, a single first mode wave in a two-layer fluid is unstable and, further, that the destabilizing disturbance realized experimentally is indeed one of those predicted above.

4. Experimental results

The experiments to be described here were performed in a lucite tank 2·5 m long, 40 cm deep and 10 cm wide, filled with a stratified fluid that was made up of layers of fresh water and salt water. The tank was initially half filled with a well-mixed solution of salt and water having a uniform density ranging from 1·07 to 1·12 g/cm^3. Fresh water was then slowly floated on to the salt water, thereby creating a fluid composed of two constant-density layers separated by a thin region in which the density varied continuously. In order to alter the experimental conditions, the thickness of this variable-density layer could be reduced by slowly withdrawing fluid from the layer, or increased by waiting for diffusion to take place. As a means of visualizing the flow, drops of toluene and carbon tetrachloride mixtures, which had been coloured with oil red dye, were injected into the upper layer and allowed to settle until they reached the level at which they were neutrally buoyant. Since the drops would then demarcate lines of constant density, they provided not only a method of visualizing the flow but also a way of determining the characteristics of the density gradient.

To produce periodic first mode internal waves, a semicircular cylinder of 5 cm radius was oscillated vertically at one end of the tank, with an amplitude that could be varied from 0·5 to 3 cm and a frequency that ranged from 0·3 to 1 c/s. Partitions were placed at the opposite end of the tank in such a way that the width of the channel gradually decreased to zero. This narrowing gap served to trap and to destroy by the action of viscosity incoming waves, thus preventing their reflexion back toward the wave-maker.

When the wave generator was first set in motion, a regular train of nearly sinusoidal first mode waves was produced. Waves of this type, an example of which is depicted in figure 2, plate 1, propagated without noticeable change of form apart from a gradual viscosity-induced decrease in amplitude. In the immediate vicinity of the wave-maker there was some mixing in the density gradient layer, but, for the most part, the motion appeared to be a nearly pure first mode internal wave.

However, if the frequency and the amplitude of the primary wave were made sufficiently large, it was found that this condition did not persist and that, as time passed, the once smooth wave became distorted by small 'lumps' in the density gradient layer. These disturbances, which propagated in the same direction as the primary wave but with a much slower speed, grew slowly in time until eventually they virtually destroyed the original motion. Figures 3, plate 1, and 4, plate 2, are photographs of this phenomenon in its initial and advanced form. As can be seen, the 'lumps' appear to be periodic in distance and, in the initial stage at least, are similar in form to a second mode periodic wave. As the instability advances, the disturbances retain their spatial periodicity and straight-crested form, even when the wave shape has become severely deformed and the centre of each 'lump' is a region of highly turbulent mixing.

In an attempt to explain this instability we first considered the possibility that the wave-maker itself produced some type of motion other than the original first mode wave. But since the frequency of the disturbance, rather than being simply related to the primary wave frequency, varied from 0·64 to 0·75 of the wave generator frequency, this explanation had to be abandoned. Similarly, the possibility that the phenomenon was a manifestation of a local shear instability was discounted both because the local Richardson number was large (on the order of 2 to 20) and because it is unlikely that such a mechanism would produce the highly periodic disturbances which were observed. Hence, we were led to propose that this phenomenon was a resonant instability of the type discussed in the previous sections.

Now, the shape of the observed disturbance, as depicted in figure 3, suggests that it is actually a second mode component of some resonantly destabilizing perturbation and, as was shown in the previous section dealing with the two-layer fluid described by $R = \operatorname{sech}^2 y$, there exists only one possible destabilizing perturbation involving a second mode wave moving in the same direction as the primary wave. This disturbance is composed of component 1, with $m_1 = 3$, moving opposite to both the primary wave and the other perturbation component with mode number $m_2 = 2$. In order to provide a test of the hypothesis that the experimentally observed 'lumps' were actually a train of second mode waves excited by this resonant interaction, we chose to make use of the fact that the analysis predicts the frequency of the excited wave. Thus, substituting the dispersion relation (15) into the resonance condition (9), we obtain a more exact version of equation (16),

$$\omega_2/\omega_0 = 0{\cdot}635+\omega_0\{0{\cdot}259\coth k_0D+0{\cdot}070\coth 0{\cdot}90\omega_0D\}+O(\omega_0^2), \tag{18}$$

where the variables are still the dimensionless ones used in the analysis. Hence, if the hypothesis about the nature of the disturbance is correct and if the density structure of the experimentally realized two-layer fluid is closely approximated by the $R(y) = \operatorname{sech}^2 y$ model, we should expect the measured disturbance frequency to be related to the primary wave frequency by means of (18).

Although, in practice, it is impossible to reproduce exactly a given density profile, measured density *vs.* height relations did not differ greatly from the idealized expression treated analytically. Furthermore, it was found earlier by Davis & Acrivos (1967) that the speed of solitary internal waves propagating in this type of stratified fluid did agree quite closely with the analytical predictions for the $\operatorname{sech}^2 y$ profile. It seemed reasonable, therefore, to expect that the propagation velocity of internal waves in the experimentally realized two-layer fluid should fall close to the values predicted theoretically for the idealized profile, and, hence, that the frequency relation (18) should provide a method for verifying our hypothesis regarding the nature of the observed instability.

The ratio ω_2/ω_0 was obtained directly by counting the number of 'lumps' passing a fixed point per unit time, and measuring the frequency of the wave generator. On the other hand, to determine the dimensionless frequency, ω_0, one had to compute first the non-dimensionalizing frequency $\Omega = (g\delta/L)^{\frac{1}{2}}$, where $\delta \equiv \frac{1}{2}\ln(\rho'_{\max}/\rho'_{\min})$ and L was chosen so that in terms of the dimensionless co-ordinate $y = \hat{y}/L$ the density gradient was given by $R = \operatorname{sech}^2 y$. The value of δ was calculated from the measured densities of the upper and lower layers, while L was obtained by measuring, in the undisturbed fluid, the elevation of neutrally buoyant drops and fitting these data to the assumed density profile. With Ω, L and the depth of the fluid known, ω_0 and D were then calculated directly and $\coth k_0D$ was found from the dispersion relation (15).

As seen in figure 5, the experimentally determined disturbance frequencies deviate only slightly from the theoretical values given by (18), thereby confirming quantitatively an important feature of our theoretical model.

Finally, it seems likely that the mixing associated with large amplitude disturbances is due to a shear instability of the type proposed by Phillips. This is so because of the large shear associated with higher mode internal waves. For example, the minimum local Richardson number, $[-g(\partial\rho/\partial y)]/(\partial u/\partial y)^2$, associated with the first mode wave in figure 2, plate 1, is of the order 15 as compared with the value $\frac{1}{8}$ estimated for the second mode disturbance in figure 4, plate 2. While the latter is only a very crude estimate, it is clear that a shear instability induced by the disturbance is a possibility.

5. Influence of viscosity

From the analysis presented, it is not clear why, of the infinite number of possible destabilizing disturbances, it is only the disturbance for which $m_1 = 3$ and $m_2 = 2$ that is observed experimentally. To give a partial answer to this question it is necessary to take into account the effects of viscosity which have been neglected up to this point, and which would be expected to affect the disturbance growth rate and hence the criteria for instability. Although, clearly, the inclusion of all viscosity effects would vastly complicate the analysis, surprisingly enough, for wave motions in fluids of the type investigated experimentally, the main results of interest can be obtained from some simple approximate calculations.

To begin, let us consider the viscous damping of a single small amplitude wave in a two-layer fluid of small viscosity, μ, and great depth. Under these conditions,

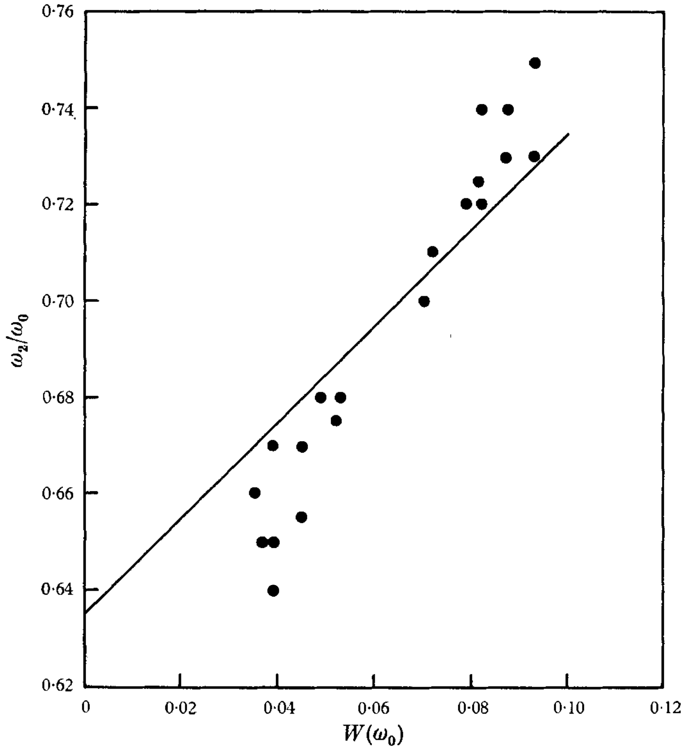


FIGURE 5. Experimentally determined frequency ratio ω_2/ω_0 against $W = \omega_0(0{\cdot}259 \coth k_0D + 0{\cdot}070 \coth 0{\cdot}90\omega_0 D)$. The solid line represents the theoretical prediction.

the velocity field will be everywhere closely approximated by the solution of the inviscid equations of motion, so that, using the solutions obtained in §2, it is possible to show (for details of the following calculations see Davis 1967) that the average total energy associated with the wave

$$\langle E\rangle = \int_0^1 dz \int_0^{2\pi/\omega} dt \int_0^{2\pi/k} dx \int_{-D}^{D} dy \{E_{\text{potential}} + E_{\text{kinetic}}\}$$

is given by

$$\langle E\rangle = \Omega^2 L^5 \frac{\pi^2}{4\omega k} a^2 \gamma^2 \int_{-D}^{D} \rho^{(0)}(y) f^2(y) R(y)\, dy,$$

where a is the amplitude of the wave. Similarly, by computing the dissipation function, we can show that the average rate of energy dissipation becomes

$$\frac{d}{dt}\langle E\rangle = -\Omega L^3 \frac{\pi^2}{4\omega k} a^2\gamma^4 \int_{-D}^{D} \mu f^2(y)\, R^2(y)\, dy.$$

Replacing μ and $\rho^{(0)}$ by their average values, and then relating the expressions for $\langle E\rangle$ and $(d/dt)\langle E\rangle$, leads finally to the equation

$$\frac{1}{a}\frac{da}{dt} \equiv -\beta = -\frac{1}{2}\left(\frac{\hat{\mu}}{\hat{\rho}}\frac{1}{\Omega L^2}\right)\gamma^2 \int_{-D}^{D} R^2 f^2\, dy \bigg/ \int_{D-}^{D} R f^2\, dy, \tag{19}$$

which determines, approximately, the rate at which a single wave is damped by viscous dissipation.

Now, assuming that the rate of energy transfer between resonantly coupled waves is, to a first approximation, independent of the fluid viscosity, we find that equations (14), which determine the amplitudes a_1 and a_2 of the disturbance components, are modified by viscous damping to

$$da_1/dt = A\alpha_1 a_2 - \beta_1 a_1, \quad da_2/dt = A\alpha_2 a_1 - \beta_2 a_2,$$

where β_1 and β_2 are defined by (19). This system admits solutions a_1 and a_2 of the form $\exp\{[-\frac{1}{2}\beta_1\beta_2 \pm (\frac{1}{4}(\beta_1-\beta_2)+A^2\alpha_1\alpha_2)^{\frac{1}{2}}]t\}$, from which it follows that exponential growth of a_1 and a_2 is possible if

$$A^2\alpha_1\alpha_2 > \beta_1\beta_2.$$

Hence we see that in a slightly viscous fluid instability occurs only if $\alpha_1\alpha_2 > 0$ and

$$A > A_{\text{critical}} \equiv (\beta_1\beta_2/\alpha_1\alpha_2)^{\frac{1}{2}}. \tag{20}$$

In order to compute this critical amplitude for the case investigated experimentally we can make use of approximate calculations presented in §3 for the $\operatorname{sech}^2 y$ density profile. First of all, for $k \ll 1$, we can calculate β_1 and β_2 from (19) using the forms for $f_i(y)$ and γ_i^2 given in (15). Similarly, we can substitute directly into (20) the value of $\alpha_1\alpha_2$ given in (16). As a result, we find that A_{critical} is related to m, the smaller of m_1 and m_2, by the relation

$$A_{\text{critical}} = \left(\frac{\hat{\mu}}{\hat{\rho}}\frac{1}{\Omega L^2}\right)\frac{2}{k_0}F(m),$$

or, in terms of dimensional variables,

$$\hat{A}_{\text{critical}} = \frac{\hat{\mu}}{\hat{\rho}}\frac{2}{\hat{k}_0}\frac{1}{(g\delta)^{\frac{1}{2}}} L^{-\frac{3}{2}} F(m), \tag{21}$$

where
$$F(m) = \frac{1}{[(m+1)^{\frac{1}{2}}-(m-1)^{\frac{1}{2}}]}\left[\frac{m^6-4m^4+4m^2-1}{m^2-\frac{9}{4}}\right]^{\frac{1}{2}},$$

and m is still greater than unity.

Since F is a monotonically increasing function of m, it is clear that the critical amplitude at which the primary wave becomes unstable to a particular disturbance is least for those perturbations composed of one second and one third mode wave, but has the same value for both of the possible disturbances made up of these waves. This fact provides a partial explanation of the nature of the experimentally observed instability, but leaves unanswered the question of why the disturbance for which $m_1 = 2$ and $m_2 = 3$ has not also been observed.

As we have noted in the previous section, the instability seemed to occur only for sufficiently large values of the primary wave amplitude and frequency, an observation which is explained by the fact that the critical amplitude is inversely proportional to k_0 and hence decreases as the frequency is increased. Similarly, as reported by Keulegan & Carpenter (1961), a wave of fixed amplitude and frequency becomes unstable when the interfacial region becomes thick, a fact which is again in qualitative agreement with (21).

It is evident then that the second-order resonant interaction analysis, when modified to account for the influence of viscous damping, serves to explain many of the experimental facts regarding the instability of oscillatory internal waves.

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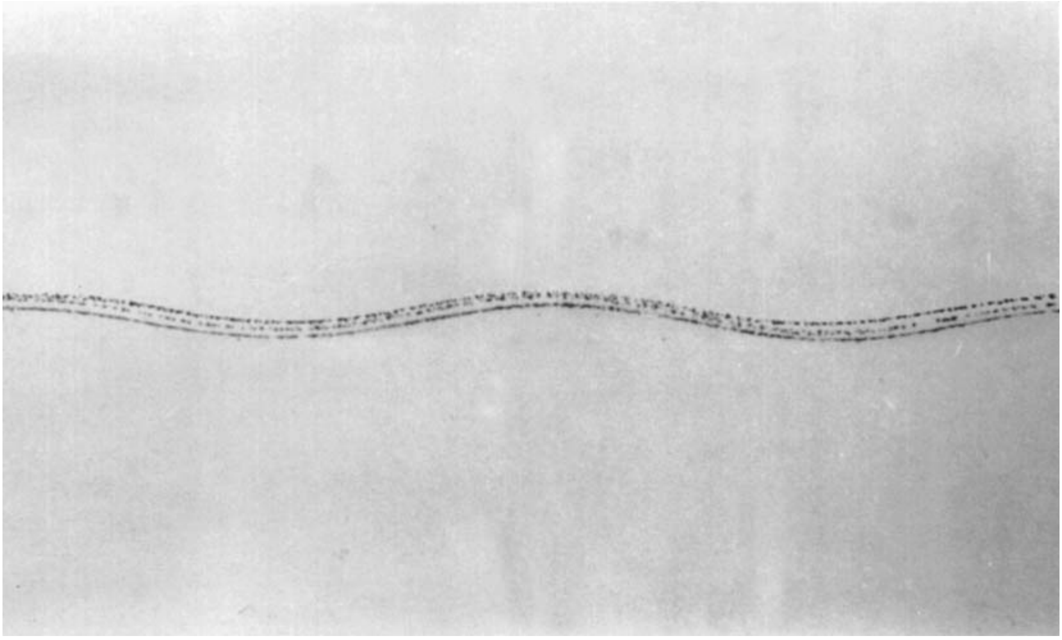

FIGURE 2

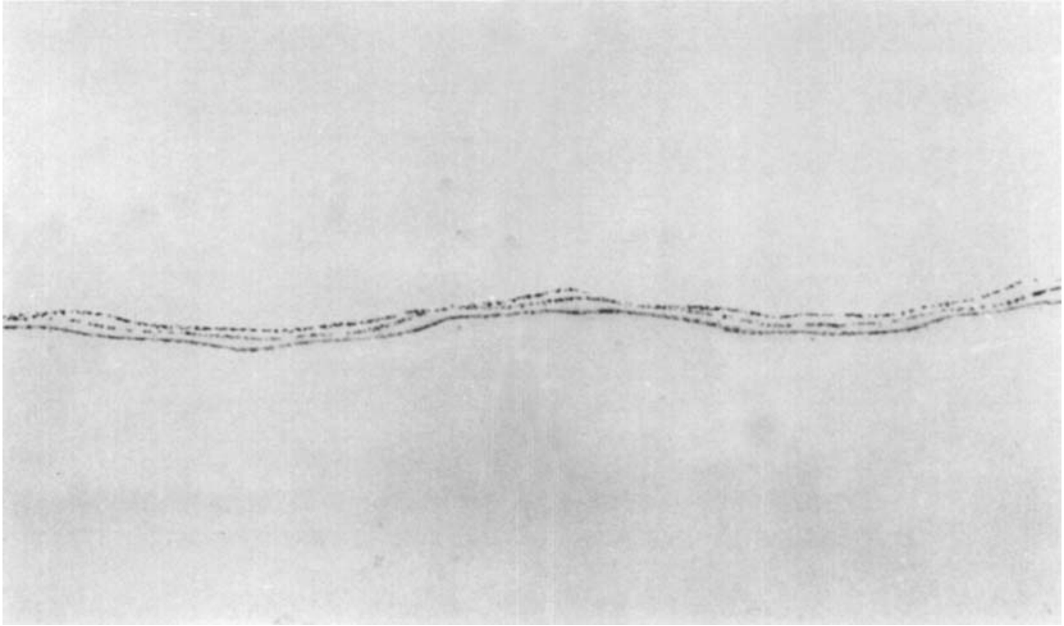

FIGURE 3

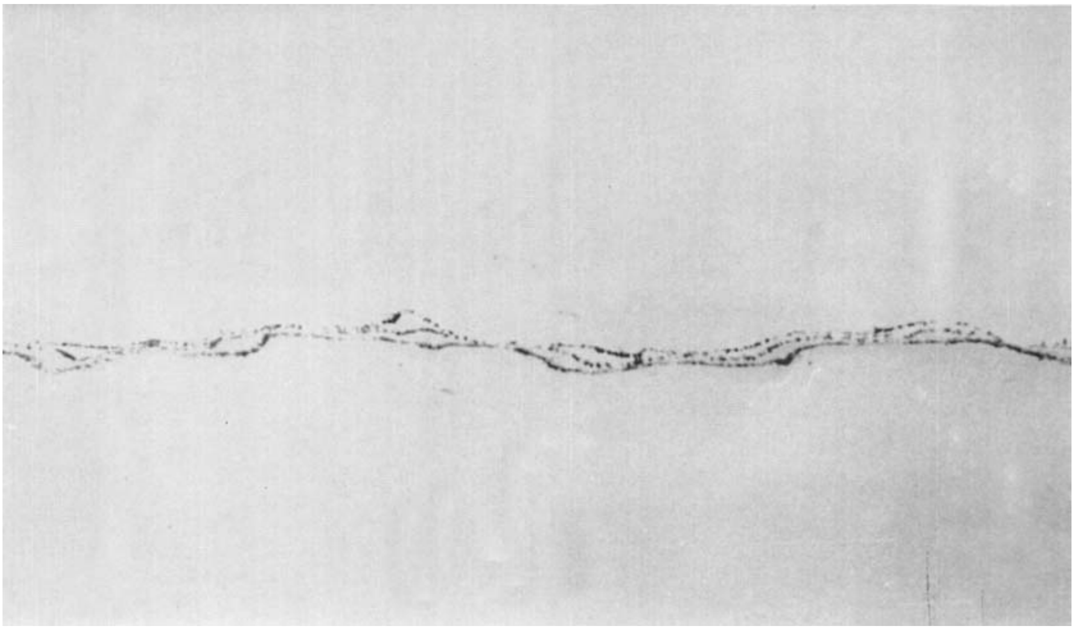

FIGURE 4